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Potts Ferromagnet: Transformations and Critical Exponents in Planar Hierarchical Lattices

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We prove that the duality transformation for a Potts ferromagnet on two-rooted planar hierarchical lattices (HL) preserves the thermal eigenvalue. This leads to a relation between the correlation length critical exponents ν of a HL and its corresponding dual lattice. Using hyperscaling, we show that their specific heat critical exponents α coincide. For a smaller class of HL—namely of diamond and tress types—we prove that another transformation also preserves ν and α.

KEY WORDS: Critical exponents; hierarchical lattices; duality.

Phase transitions of the q-state Potts model on hierarchical lattices (HL) have been largely studied with real-space renormalization group methods because exact calculations can be performed on such lattices.[1-5] More recently Hu[6] and da Silva and Tsallis[7] obtained some intriguing results studying critical properties of Ising and Potts ferromagnets on HLs. Hu[6] exhibits two different HLs (see below, Figs. 1a and 1c) that present the same thermal eigenvalue λ. Da Silva and Tsallis[7] show that generalized diamond and tress HLs (see examples in Fig. 1 below) have the same correlation length critical exponents ν. We will show that these results are consequences of two HL transformations rather than singular cases. The first one is the duality,[2] a property of *any* planar HL. The second is related to a smaller class of HLs, namely the generalized diamond and tress ones[7,8]; it transforms a diamond HL into a stress HL and conversely.

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In order to show the properties induced by these transformations, we will consider a q-state Potts ferromagnet on a HL. The Hamiltonian is given by

$$\mathscr{H} = -qJ \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}, \qquad \sigma_i = 0, 1, \ldots, q-1 \tag{1}$$

where the sum is over nearest neighbor sites and δ is the Kronecker delta. The σ variables are on the sites of the HL and the coupling constants are associated with the bonds. We will use a very convenient variable, the thermal transmissivity

$$t \equiv [1 - \exp(-qJ/k_B T]/[1 + (q-1)\exp(-qJ/k_B T)]$$

associated with each bond of the HL.[9] Its dual variable τ is defined by the relation[9]

$$\tau \equiv \frac{1-t}{1+(q-1)t} \tag{2}$$

The recursive relation of a two-rooted graph corresponding to the HL basic cell with length b and aggregation number A[3] is given by $t' = G(t)$, where $G(t)$ is a ratio of two polynomials of t.[10,11] The thermal eigenvalue of this HL is given by

$$\lambda \equiv \left.\frac{\partial G}{\partial t}\right|_{t^*} \tag{3}$$

where t^* satisfies $t^* = G(t^*)$.

Considering that, for HLs whose basic cells are two-rooted planar graphs, the function G associated with a HL is related to the function $\tilde{G}$ associated with the dual HL by the equation[10]

$$G(t) = \frac{1 - \tilde{G}(\tau)}{1 + (q-1)\tilde{G}(\tau)} \tag{4}$$

we are able to prove the following property.

Property 1. The Potts thermal eigenvalues of a two-rooted planar HL and of its dual lattice coincide.

The proof is straightforward. We must take the derivative of Eq. (4) with respect to t, then use the chain rule on the right-hand side [having in mind that τ is related to t by Eq. (2)] and evaluate the derivatives at the fixed point t^*. Considering that $\tau^* = \tau(t^*)$ and $\tilde{G}(\tau^*) = \tau^*$, we verify that

$$\left.\frac{\partial G}{\partial \tilde{G}}\right|_{\tilde{G}=\tau^*} = \left(\left.\frac{d\tau}{dt}\right|_{t^*}\right)^{-1}$$

This leads to the equality between the thermal eigenvalues of the HLs associated with G and $\tilde{G}$ (dual)

$$\lambda = \tilde{\lambda} \tag{5}$$

where λ is given by Eq. (3) and $\tilde{\lambda} = (\partial\tilde{G}/\partial\tau)|_{\tau^*}$.

Corollary 1. With b the basic cell minimum length of a HL and $\tilde{b}$ the corresponding length of its dual lattice, Eq. (5) can be written as

$$b^{1/\nu} = \tilde{b}^{1/\tilde{\nu}} \tag{6}$$

where ν and $\tilde{\nu}$ are the correlation length critical exponents of a HL and its dual lattice, respectively. The definition of intrinsic dimension,[3] namely $D \equiv \log A/\log b$, and the fact that duality transformation preserves the aggregation number A enable us to rewrite Eq. (6) as

$$D\nu = \tilde{D}\tilde{\nu} \tag{7}$$

Corollary 2. Using the hyperscaling relation for a HL,[4] $D\nu = 2-\alpha$, we have

$$\alpha = \tilde{\alpha} \tag{8}$$

thus showing that the specific heat critical exponents of a HL and of its dual lattice are the *same*. We remark that the relations between critical exponents given by Eqs. (7) and (8) are valid for *all* planar HL. Furthermore, if $b = \tilde{b}$, then also $\nu = \tilde{\nu}$.

The second property is related to a smaller class of planar HL. This class is partitioned into two subclasses, namely the diamondlike and tresslike HLs. The basic cell of a diamong HL is made up of N branches in parallel, each one with b bonds in series, and the tress basic cell is made up of b clusters in series, each one with N bonds in parallel. For example, the basic cells of Figs. 1a and 1c generate diamond HLs with $b=2$, $N=3$ and $b=3$, $N=2$, respectively, and those of Figs. 1b and 1d generate tress HLs with $b=3$, $N=2$ and $b=2$, $N=3$ respectively. The expression for G_D (G_T) of the diamond (tress) HL for any b and N is given by

$$G_D(t, b, N) = \frac{1-\{(1-t^b)/[1+(q-1)t^b]\}^N}{1+(q-1)\{(1-t^b)/[1+(q-1)t^b]\}^N} \tag{9}$$

$$G_T(t, b, N) = \left[\frac{1-\{(1-t)/[1+(q-1)t]\}^N}{1+(q-1)\{(1-t)/[1+(q-1)t]\}^N}\right]^b \tag{10}$$

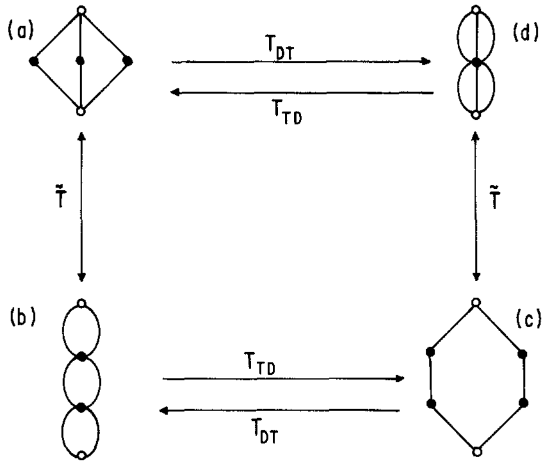

Fig. 1. (a, c) Diamond and (b, d) tress HL basic cells connected by the transformations $\tilde{T}$, T_{DT}, and T_{TD}: (○) roots, (●) internal sites.

It is easy to verify by Eqs. (9) and (10) that there is a relation between G_D and G_T given by

$$G_T(\omega, b, N) = [G_D(t, b, N)]^b \tag{11}$$

where $\omega = t^b$. Thus, the diamond and tress HL can be connected by two transformations, the diamond–tress (T_{DT}), given by

$$T_{DT}: \quad G_D(t, b, N) \to [G_D(t, b, N)]^b = G_T(t^b, b, N) \tag{12}$$

and its inverse (tress–diamond T_{TD})

$$T_{TD}: \quad G_T(t, b, N) \to [G_T(t, b, N)]^{1/b} = G_D(t^{1/b}, b, N) \tag{13}$$

where the equalities in (12) and (13) follow from Eq. (11). Clearly, T_{DT} and T_{TD} are related to the diamond–tress transformations proposed by Ottavi and Albinet.[8]

Property 2. A diamondlike and a tresslike HL with the same b and N share the same Potts correlation length critical exponent ν.

This proof is straightforward. We take the derivative of Eq. (11) with respect to t and evaluate this derivative at the critical point t^*. Having in mind that $\omega^* = \omega(t^*) = t^{*b}$, this leads to $\lambda_T = \lambda_D$. Since b is the same for diamond and tress HLs connected by T_{DT}, this implies that $\nu_T = \nu_D$.

Corollary. Since the diamond and tress HLs connected by T_{DT} (or T_{TD}) have the same b, this implies that their intrinsic dimensionalities are the same. By using hyperscaling, it follows that their specific heat critical exponents are the same, $\alpha_T = \alpha_D$.

It is worthwhile to note that if $b \neq N$ ($D \neq 2$), then the tress HL is not the dual of a diamond HL. Also, if $b = N$ ($D = 2$), then the T_{DT} transformation turns out to be the duality transformation.

For diamond and tress HLs the conjugation of the transformations defined by Eqs. (4) and (11), namely $\tilde{T}$ and T_{DT}, connects four HLs, as illustrated in Fig. 1.

In conclusion, the HLs connected by $\tilde{T}$ share the same α and their correlation length critical exponents are related by $D\nu = \tilde{D}\tilde{\nu}$; the HLs connected by T_{DT} have both α and ν equal.

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